LIBRARY OF THE
UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN
510.84
Il6r
no.111-130
cop.3

The person charging this material is responsible for its return to the library from which it was withdrawn on or before the **Latest Date** stamped below.

Theft, mutilation, and underlining of books are reasons for disciplinary action and may result in dismissal from the University.
To renew call Telephone Center, 333-8400

UNIVERSITY OF ILLINOIS LIBRARY AT URBANA-CHAMPAIGN

BUILDING USE ONLY
SEP 21 1980

SEP 20 1980

L161—O-1096

Digitized by the Internet Archive
in 2013

http://archive.org/details/conservationlaws114taub

I l6r
no. 114
cop. 3

CONSERVATION LAWS AND VARIATIONAL PRINCIPLES IN GENERAL RELATIVITY*

by

A. H. Taub

UNIVERSITY OF ILLINOIS
DIGITAL COMPUTER LABORATORY
URBANA, ILLINOIS

Report No. 114

April 23, 1962

---

* This work was supported in part by the National Science Foundation under Grant G16489.

ABSTRACT

Special relativistic field theories involving tensor fields and derivable from variational principles are generalized to general relativistic theories. It is assumed that the Lagrangian is a function of the tensor field components and their first derivatives. The Lagrangian is first written in a general coordinate system in Minkowski space-time and then the restriction that the underlying space-time is flat is removed. The Einstein field equations for the gravitational field may be derived as was shown by Hilbert, and the matter energy tensor is obtained as a function of the tensor field. The technique of E. Noether is used to derive conservation laws and relations between the various matter energy tensors which arise.

CONSERVATION LAWS AND VARIATIONAL PRINCIPLES IN GENERAL RELATIVITY

1. Introduction

In this paper we examine a method of formulating in general relativity a field theory described in special relativity by a variational principle involving a Lagrangian function which is a scalar function, in Minkowski space time, and which in turn depends on a tensor field $\varphi^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$ (written as φ^A) and its first derivatives. The method we shall employ is derived from a form of the equivalence principle and is as follows:

We shall use a general curvalinear coordinate system in Minkowski space-time in which to express the Lagrangian function $\mathcal{L}$. We shall then consider the $g_{\mu\nu}$ occurring in $\mathcal{L}$ as the metric tensor of a general Riemannian space and determine the nature of this space by deriving the Einstein field equations from the variational principle

$$\delta_g I = 0 \tag{1.1}$$

with

$$I = \int_V (R + K\mathcal{L}) \sqrt{-g}\, d^4x, \tag{1.2}$$

R the scalar curvature of the Riemannian space, and $\delta_g I$ the variation in I produced by a variation of the quantities $g_{\mu\nu}$. The equation determining the tensor field φ^A will be taken to be the Euler equation of

$$\delta_\varphi I = 0 \tag{1.3}$$

where $\delta_\varphi I$ is the variation in I, given by equation (1.2), produced by a variation of the tensor field $\varphi^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$.

It will be shown that there exists a symmetric tensor $T^{\mu\nu}$ formed from the components of φ^A and $g_{\mu\nu}$ which as a consequence of equation (1.3) satisfies

$$T^{\mu\nu}_{\ ;\nu} = 0 \qquad (1.4)$$

where the semi-colon denotes the covariant derivative and as a consequence of (1.2) satisfies

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\kappa T^{\mu\nu} \qquad (1.5)$$

where $R^{\mu\nu}$ is the Ricci tensor of the space-time with metric tensor $g_{\mu\nu}$. The tensor $T^{\mu\nu}$ may thus be said to represent the "gravitational matter tensor" which "creates" the gravitational field represented by the metric tensor $g_{\mu\nu}$.

Another tensor $t^{\mu\nu}$, a nonsymmetric one, which is a generalization of the special relativity "inertial" stress-energy tensor of the field φ^A will also be derived. It will be shown that

$$t^{\rho\mu} + 2T^{\rho\mu} + N^{\rho\mu\lambda}_{\ ;\lambda} = 0 \qquad (1.6)$$

where

$$N^{\rho\mu\lambda} = - N^{\rho\lambda\mu} \qquad (1.7)$$

Equation (1.6) thus relates the inertial and gravitational matter tensors.

2. Notation

In order to avoid an excessive use of indices, we employ the following notation.

$$\varphi^A \sim \varphi^{\sigma_1 \ldots \sigma_n}{}_{\tau_1 \ldots \tau_m} \quad . \qquad (2.1)$$

The symbol ~ is to be read as "stands for."

$$\varphi^A{}_{;\mu} \sim \varphi^{\sigma_1 \ldots \sigma_n}{}_{\tau_1 \ldots \tau_m ;\mu} = \varphi^{\sigma_1 \ldots \sigma_n}{}_{\tau_1 \ldots \tau_m ,\mu}$$

$$+ \sum_i \varphi^{\sigma_1 \ldots \sigma_{i-1} \rho \sigma_{i+1} \ldots \sigma_n}{}_{\tau_1 \ldots \tau_m} \Gamma^{\sigma_i}_{\rho\mu} \qquad (2.2)$$

$$- \sum_j \varphi^{\sigma_1 \ldots \sigma_n}{}_{\tau_1 \ldots \tau_{j-1} \rho \tau_{j+1} \ldots \tau_m} \Gamma^{\rho}_{\tau_j \mu}$$

where the comma denotes the ordinary derivative and

$$\Gamma^{\rho}_{\sigma\tau} = g^{\rho\lambda} \frac{1}{2} (g_{\sigma\lambda,\tau} + g_{\lambda\tau,\sigma} - g_{\sigma\tau,\lambda}). \tag{2.3}$$

We also write ψ_A, and ψ^{μ}_{A} where

$$\psi_A \sim \psi^{\tau_1 \dots \tau_m}{}_{\sigma_1 \dots \sigma_n}, \tag{2.4}$$

$$\psi^{\mu}_{A} \sim \psi^{\tau_1 \dots \tau_m}{}_{\sigma_1 \dots \sigma_n}{}^{\mu}$$

and

$$\psi_A \varphi^A \sim \psi^{\tau_1 \dots \tau_m}{}_{\sigma_1 \dots \sigma_n} \varphi^{\sigma_1 \dots \sigma_n}{}_{\tau_1 \dots \tau_m} \tag{2.5}$$

$$\psi^{\mu}_{A} \varphi^A \sim \psi^{\tau_1 \dots \tau_m}{}_{\sigma_1 \dots \sigma_n}{}^{\mu} \varphi^{\sigma_1 \dots \sigma_n}{}_{\tau_1 \dots \tau_m} \tag{2.6}$$

That is, the quantity given in (2.5) is a scalar and that in (2.6) is a vector.

We consider $\mathcal{L}$ as a function of three sets of variables: $g_{\mu\nu}$, φ^A and $\varphi^A{}_{;\mu}$ and define

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + \frac{1}{2} g^{\mu\nu} \mathcal{L} \tag{2.7}$$

$$q_A = \frac{\partial \mathcal{L}}{\partial \varphi^A} \tag{2.8}$$

$$p^{\mu}_{A} = \frac{\partial \mathcal{L}}{\partial \varphi^A{}_{;\mu}} . \tag{2.9}$$

In each case the remaining two sets of variables are kept constant in the partial differentiations.

We also define

$$P^{\rho\lambda\mu} = \sum_i p^{\tau_1 \dots \tau_m}{}_{\sigma_1 \dots \sigma_n}{}^{\mu} \varphi^{\sigma_1 \dots \sigma_{i-1} \lambda \sigma_{i+1} \dots \sigma_n}{}_{\tau_1 \dots \tau_m} g^{\sigma_i \rho} - \sum_j p^{\tau_1 \dots \tau_{j-1} \lambda \tau_{j+1} \dots \tau_m}{}_{\sigma_1 \dots \sigma_n}{}^{\mu} \varphi^{\sigma_1 \dots \sigma_n}{}_{\tau_1 \dots \tau_{m-1} \sigma \tau_j \dots \tau_m} g^{\sigma\rho} \tag{2.10}$$

It follows from equations (2.3) that

$$\delta\Gamma^{\rho}_{\sigma\tau} = g^{\rho\lambda}\,\frac{1}{2}\,(\delta g_{\sigma\lambda;\tau} + \delta g_{\lambda\tau;\sigma} - \delta g_{\sigma\tau;\lambda}) \tag{2.11}$$

and from (2.2) that

$$\delta(\varphi^{A}{}_{;\mu}) - (\delta\varphi^{A})_{;\mu} \sim \sum_{i} \varphi^{\sigma_1\cdots\sigma_{i-1}\rho\sigma_{i+1}\cdots\sigma_n}{}_{\tau_1\cdots\tau_m}\,\delta\Gamma^{\sigma_i}_{\rho\mu}$$
$$- \sum_{j} \varphi^{\sigma_1\cdots\sigma_m}{}_{\tau_1\cdots\tau_{j-1}\rho\tau_{j+1}\cdots\tau_m}\,\delta\Gamma^{\rho}_{\tau_j\mu}$$

where the variations are produced by varying the φ^{A} and the $g_{\mu\nu}$.

Hence

$$\begin{aligned} p^{\mu}_{A}\,\delta(\varphi^{A}{}_{;\mu}) - p^{\mu}_{A}\,(\delta\varphi^{A})_{;\mu} &= \frac{1}{2}\,P^{\lambda\rho\mu}\,(\delta g_{\lambda\rho;\mu} + \delta g_{\lambda\mu;\rho} - \delta g_{\mu\rho;\lambda}) \\ &= M^{\lambda\rho\mu}\cdot\,\delta g_{\lambda\rho;\mu} \end{aligned} \tag{2.12}$$

where

$$M^{\lambda\rho\mu} = \frac{1}{4}\,(P^{\lambda\rho\mu} + P^{\rho\lambda\mu} + P^{\lambda\mu\rho} + P^{\rho\mu\lambda} - P^{\mu\rho\lambda} - P^{\mu\lambda\rho}) \tag{2.13}$$

It follows from this equation that

$$\begin{aligned} 2M^{\lambda\rho\mu} - P^{\lambda\rho\mu} &= \frac{1}{2}\,(P^{\rho\lambda\mu} - P^{\lambda\rho\mu} + P^{\lambda\mu\rho} + P^{\rho\mu\lambda} - P^{\mu\rho\lambda} - P^{\mu\lambda\rho}) \\ &\equiv N^{\lambda\rho\mu} = -\,N^{\lambda\mu\rho} \end{aligned} \tag{2.14}$$

3. The Euler Equations

The relation between the energy-momentum tensor for the φ-field which appears in the Einstien field equations and the variation of $\mathcal{L}$ with respect to the $g_{\mu\nu}$ was first pointed out by Hilbert[1] as is noted in Pauli's classical discussion of the theory of relativity [2] where additional references may be found. In this section we shall derive the Einstein field equations and the equations that must be satisfied by the φ-field.

The total variation of the integral I given by equation (1.2), that is the change in I produced by the changes $\delta g_{\mu\nu}$ and $\delta\varphi^A$ is given by

$$\begin{aligned}\delta I &= \delta_g I + \delta_\varphi I \\ &= -\int_V \Big[+ (G^{\mu\nu} + \kappa\theta^{\mu\nu})\,\delta g_{\mu\nu} + \kappa(q_A\delta\varphi^A + p^\mu_A\,\delta(\varphi^A_{;\mu})) \\ &\qquad - \Big\{(g^{\rho\sigma}\,g^{\mu\nu} - g^{\rho\mu}g^{\nu\sigma})\delta g_{\mu\nu;\rho}\Big\}_{;\sigma}\Big]\sqrt{-g}\;d^4x \end{aligned} \tag{3.1}$$

where $G^{\mu\nu}$ is defined in equations (1.5), q_A in (2.8) and p^μ_A in equation (2.9).

It follows from equation (2.12) that equation (3.1) may be written as

$$\begin{aligned}\delta I = \int_V \Big[&(G^{\mu\nu} + \kappa\theta^{\mu\nu})\,\delta g_{\mu\nu} + \kappa(q_A\delta\varphi^A + p^\mu_A\,(\delta\varphi^A)_{;\mu} \\ &+ M^{\mu\nu\sigma}\,\delta g_{\mu\nu;\sigma}) \\ &- \Big\{\,(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\mu}g^{\nu\sigma})\;\delta g_{\mu\nu;\rho}\Big\}_{;\sigma}\Big]\sqrt{-g}\;d^4x\end{aligned}$$

On integrating by parts this in turn may be written as

$$\begin{aligned}\delta I = &-\int_V \Big\{\Big[G^{\mu\nu} + \kappa T^{\mu\nu}\Big]\,\delta g_{\mu\nu} + \kappa(q_A - p^\mu_{A;\mu})\;\delta\varphi^A\Big\}\sqrt{-g}\;d^4x \\ &-\int_V \Big[\kappa M^{\mu\nu\sigma}\;\delta g_{\mu\nu} + \kappa p^\sigma_A\,\delta\varphi^A - (g^{\rho\sigma}g^{\mu\nu} - g^{\rho\mu}g^{\nu\sigma})\delta g_{\mu\nu;\rho}\Big]_{;\sigma}\sqrt{-g}\;d^4x\end{aligned} \tag{3.2}$$

where the symmetric tensor

$$T^{\mu\nu} = \theta^{\mu\nu} - M^{\mu\nu\sigma}_{\;\;;\sigma} = T^{\nu\mu} \tag{3.3}$$

and $M^{\mu\nu\sigma}$ is defined by equations (2.10) and (2.13).

By requiring δI to vanish for arbitrary variations which vanish on the boundary of the region V we obtain the Euler equations

$$G^{\mu\nu} + K T^{\mu\nu} = 0 \tag{3.4}$$

and

$$F_A \equiv q_A - p^{\mu}_{A;\mu} = 0 \tag{3.5}$$

Equations (3.4) are the Einstein field equations with a matter tensor given by equation (3.3). Because this tensor arises from the variation of I with respect to the gravitational field $g_{\mu\nu}$ we call $T^{\mu\nu}$ the gravitational matter tensor. Note that even in the Minkowski space-time $T^{\mu\nu}$ is different from $\theta^{\mu\nu}$ in a general coordinate system.

Equations (3.5) are the equations for the field φ^A. They may be obtained from the special relativity equations in a galilean coordinate system by replacing every ordinary derivative by a covariant one. They obviously reduce to the equations of special relativity in case the tensor $g_{\mu\nu}$ is the metric tensor of Minkowski space-time.

4. Infinitesimal Coordinate Transformations

In this section we shall use a technique similar to that of E. Noether [3] to derive conservation laws from the invariance properties of the Lagrangian.

Under the infinitesimal coordinate transformation

$$x^{\sigma} \rightarrow x^{\sigma *} = x^{\sigma} + \xi^{\sigma} \tag{4.1}$$

where ξ^{σ} is an arbitrary vector, the products of whose component may be neglected, the integral I undergoes the variation

$$\delta I = - \int_V \left[(R - K \mathcal{L}) \xi^{\sigma} \right]_{;\sigma} \sqrt{-g}\, d^4x \tag{4.2}$$

since the boundary of the region of integration is transformed and since the integrand is a scalar function of coordinates.

Since the $g_{\mu\nu}$ and φ^A are tensors they are transformed under (4.1) in accordance with the equations

$$\overset{*}{g}_{\mu\nu}(\overset{*}{x}) = g_{\sigma\tau}\,(x(\overset{*}{x}))\,\frac{\partial x^\sigma}{\partial \overset{*}{x}{}^\mu}\,\frac{\partial x^\tau}{\partial \overset{*}{x}{}^\nu}$$

and a corresponding equation for φ^A. These produce variations in the fields $g_{\mu\nu}$ and φ^A given by

$$\delta g_{\mu\nu} = -\,\xi_{\mu;\nu} - \xi_{\nu;\mu} \tag{4.3}$$

$$\begin{aligned} \delta\varphi^A \sim -\,\varphi^{\sigma_1\ldots\sigma_n}{}_{\tau_1\ldots\tau_m;\rho}\,\xi^\rho &+ \sum_i \varphi^{\sigma_1\ldots\sigma_{i-1}\rho\sigma_{i+1}\ldots\sigma_n}{}_{\tau_1\ldots\tau_m}\,\xi^{\sigma_i}{}_{;\rho} \\ &- \sum_j \varphi^{\sigma_1\ldots\sigma_n}{}_{\tau_1\ldots\tau_{j-1}\rho\tau_{j+1}\ldots\tau_m}\,\xi^\rho{}_{;\tau_j} \end{aligned} \tag{4.4}$$

On substituting equations (4.3) and (4.4) into equation (3.2) we obtain

$$\begin{aligned} \delta I = &\int_V \left\{ (2K T^{\mu\nu} + K L^{\mu\nu})\,\xi_{\mu;\nu} + K F_A \varphi^A{}_{;\rho}\,\xi^\rho \right\} \sqrt{-g}\, d^4x \\ &+ K \int_V (N^{\mu\nu\sigma}\,\xi_{\mu;\nu} + t_\rho{}^\sigma \xi^\rho)_{;\sigma} \sqrt{-g}\, d^4x \\ &- \int_V \Big[(R - K \mathscr{L})\,\xi^\sigma \Big]_{;\sigma} \sqrt{-g}\, d^4x \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} L^{\mu\nu} = &-\sum_i F^{\tau_1\ldots\tau_m}{}_{\sigma_1\ldots\sigma_{i-1}\sigma\sigma_{i+1}\ldots\sigma_n}\, g^{\sigma\mu}\, \varphi^{\sigma_1\ldots\sigma_{i-1}\nu\sigma_{i+1}\ldots\sigma_n}{}_{\tau_1\ldots\tau_m} \\ &+ \sum_j F^{\tau_1\ldots\tau_{j-1}\nu\tau_{j+1}\ldots\tau_m}{}_{\sigma_1\ldots\sigma_n}\, \varphi^{\sigma_1\ldots\sigma_n}{}_{\tau_1\ldots\tau_{j-1}\rho\tau_{j+1}\ldots\tau_m}\, g^{\rho\mu} \end{aligned} \tag{4.6}$$

and

$$t_\rho{}^\sigma = p_A{}^\sigma\, \varphi^A{}_{;\rho} - \delta^\sigma_\rho \mathscr{L}; \tag{4.7}$$

in deriving (4.5) we have made use of the fact that

$$2G^{\mu\nu}\xi_{\mu;\nu} - (g^{\rho\sigma}g^{\mu\nu}-g^{\rho\mu}g^{\nu\sigma})(\xi_{\mu;\nu}+\xi_{\nu;\mu})_{;\rho\sigma} = -\ (R\xi^{\sigma})_{;\sigma}$$

Note that

$$L^{\mu\nu} = 0$$

when equations (3.5) are satisfied.

Since the tensor $N^{\mu\nu\sigma}$ is antisymmetric in ν and σ we have

$$(\xi_{\mu}N^{\mu\nu\sigma})_{;\nu\sigma} = 0$$

That is,

$$(\xi_{\mu;\nu}N^{\mu\nu\sigma} + \xi_{\mu}N^{\mu\nu\sigma}{}_{;\nu})_{;\sigma} = 0$$

or

$$(\xi_{\mu;\nu}N^{\mu\nu\sigma})_{;\sigma} = (\xi_{\mu}N^{\mu\sigma\nu}{}_{;\nu})_{;\sigma}$$

In view of this equation and equation (4.2) we may write equation (4.5) as

$$\int_V \left[(2T^{\mu\nu} + L^{\mu\nu})\ \xi_{\mu;\nu} + F_A\varphi^A{}_{;\rho}\xi^{\rho}\right]\sqrt{-g}\ d^4x$$
$$+\int_V \left[(t_{\rho}{}^{\sigma} + N_{\rho}{}^{\sigma\nu}{}_{;\nu})\ \xi^{\rho}\right]_{;\sigma}\sqrt{-g}\ d^4x = 0 \qquad (4.8)$$

On integrating this equation by parts we obtain

$$\int_V (F_A\varphi^A{}_{;\rho} - 2T_{\rho}{}^{\sigma}{}_{;\sigma} - L_{\rho}{}^{\sigma}{}_{;\sigma})\ \xi^{\rho}\sqrt{-g}\ d^4x$$
$$+\int_V \left[(2T_{\rho}{}^{\sigma} + t_{\rho}{}^{\sigma} + N_{\rho}{}^{\sigma\nu}{}_{;\nu})\ \xi^{\rho}\right]_{;\sigma}\sqrt{-g}\ d^4x = 0 \qquad (4.9)$$

When the tensor field φ^A is a solution of the Euler equations (3.5) the above equations become

$$2\int_V T^{\mu\nu}\xi_{\mu;\nu}\sqrt{-g}\ d^4x + \int_V \left[(t_{\rho}{}^{\sigma} + N_{\rho}{}^{\sigma\nu}{}_{;\nu})\xi^{\rho}\right]_{;\sigma}\sqrt{-g}\ d^4x \qquad (4.10)$$

and

$$- 2 \int_V T^{\mu\nu}{}_{;\nu} \, \xi_\mu \sqrt{-g} \, d^4x + \int_V \left[(2T_\rho{}^\sigma + t_\rho{}^\sigma + N_\rho{}^{\sigma\nu}{}_{;\nu}) \xi^\rho \right]_{,\sigma} \sqrt{-g} \, d^4x = 0 \qquad (4.11)$$

Both equations must hold for arbitrary volumes and arbitrary vectors ξ_μ.

If ξ_μ is a Killing vector, that is, satisfies

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

then equations (4.10) implies for such vectors that

$$\left[(t_\rho{}^\sigma + N_\rho{}^{\sigma\nu}{}_{;\nu}) \xi^\rho \right]_{;\sigma} = \left[t_\rho{}^\sigma \xi^\rho + N^{\rho\nu\sigma} \xi_{\rho;\nu} \right]_{;\sigma} = 0 \qquad (4.12)$$

Killing vectors need not exist in a space-time satisfying the Einstein field equations. As is well known there are ten linearly independent Killing vectors in Minkowski space-time--the generators of the inhomogeneous Lorentz group. Equations (4.12) for these Killing vectors are the conservation laws of energy momentum and angular momentum discussed by Belinfante and by Rosenfeld. Since the tensor $t^{\rho\sigma}$ satisfies these conservation laws, in special relativity, we may consider it as the inertial energy tensor.

Since equation (4.11) must hold for arbitrary vectors and arbitrary volumes, we must have

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad (4.13)$$

and

$$2T^{\rho\sigma} + t^{\rho\sigma} + N^{\rho\sigma\nu}{}_{;\nu} = 0 \qquad (4.14)$$

Equation (4.13) is a consequence of equation (3.4). We have now shown that it follows from the invariance properties of $\mathcal{L}$ and the φ^A field equations, equation (3.5). Thus equations (4.13) and (4.14) hold in special relativity.

As was pointed out in the introduction, equation (4.14) relates two energy tensors, the gravitational one $T^{\mu\nu}$ and the inertial one $t^{\mu\nu}$.

It follows from equations (4.13) and (4.14) that

$$t^{\rho\sigma}{}_{;\sigma} = - N^{\rho\sigma\nu}{}_{;\nu\sigma} = - \frac{1}{2} (N^{\rho\sigma\nu}{}_{;\nu\sigma} - N^{\rho\sigma\nu}{}_{;\sigma\nu})$$

$$= \frac{1}{2} N^{\tau\sigma\nu} R^{\rho}{}_{\tau\nu\sigma} \tag{4.15}$$

This equation, which may be derived directly from the definition of $t^{\rho\sigma}$ and equations (3.5), reduces to

$$t^{\rho\sigma}{}_{;\sigma} = 0$$

in the case of special relativity.

It should be noted that as a consequence of equation (4.13), equation (4.11) may be written as

$$\int_V \left[(2T_\rho{}^\sigma + t_\rho{}^\sigma + N_\rho{}^{\sigma\nu}{}_{;\nu}) \, \xi^\rho \right]_{;\sigma} \sqrt{-g} \, d^4x = 0$$

which in turn may be written as an integral over the hypersurface S boundary the volume V, namely

$$\int_S \left[2T_\rho{}^\sigma + t_\rho{}^\sigma + N_\rho{}^{\sigma\nu}{}_{;\nu} \right] \xi^\rho \, n_\sigma dS = 0 \tag{4.16}$$

where $n_\sigma dS$ is the element of volume in S, and ξ^ρ is an arbitrary vector. Equation (4.13) may be used to relate the time rate of change of the three-demensional volume integrals of T_ρ^4 and t_ρ^4 by choosing the hypersurface surface S to consist of the hyperplanes t = constant and t + dt = constant.

5. Concluding Remarks

The results obtained above may be readily generalized to the case where there are a number of φ-fields present. In such a case for each such field there will be an associated $T^{\mu\nu}$ and a corresponding $t^{\mu\nu}$. The right-hand side of the Einstein field equations will contain the sum of the $T^{\mu\nu}$

and this tensor will be related to the sum of the $t^{\mu\nu}$ by equations analogous to equations (4.14).

In case the φ-field is a spinor field a simular discussion to that given above can be made. The special relativistic Lagrangian must first be generalized by replacing ordinary derivatives of the spinor field by covariant ones. The variations in the metric tensor may be performed by varying the generalized Dirac matrices which satisfy the relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}\, 1.$$

The details of this process will be described in a subsequent paper.

It should be pointed out that if we define the scalar

$$\mathcal{H} = p_A^\sigma \varphi^A_{;\sigma} - \mathcal{L}$$

then equations (3.5) become

$$p_{A;\sigma}^{\sigma} = - \frac{\partial \mathcal{H}}{\partial \varphi^A} \qquad (5.1)$$

and

$$\varphi^A_{;\sigma} = \frac{\partial \mathcal{H}}{\partial p_A^\sigma} \qquad (5.2)$$

These two equations are similar to the Hamiltonian equations for particles.

Note, however, that $\mathcal{H}$ is not $t_\rho^{\ \rho}$ nor $t_4^{\ 4}$. Thus if in virtue of equations (5.1) we call $\mathcal{H}$ the Hamiltonian it is related to $t_\rho^{\ \sigma}$ through the equations

$$\mathcal{H} = t_\rho^{\ \rho} + 3\mathcal{L} \qquad (5.3).$$

Thus the connection of the Hamiltonian with the stress energy tensor involves the Lagrangian function.

References

1. D. Hilbert, Nachr. Ges. Wiss. Göttingen, p. 395 (1915).
2. W. Pauli, Theory of Relativity, Pergamon Press, London, p. 158, (1958).
3. E. Noether, Göttingen Nachtrichten, p. 235, (1918).
4. L. Rosenfeld, Acad. Roy. Belgique, 18, p. 6, (1940).
5. F. J. Belinfante, Physics 7, p. 887 (1939).

BOUND BY
HNM

UNIVERSITY OF ILLINOIS-URBANA
510.84 IL6R v.1 C002 v.111-130(1961)
Some memory elements used in ILLIAC II /
3 0112 088404147